

Developing Analytic Formulae of Risk and Reward Measures for Discrete-Time CPPI with Regime Switching and Transaction Cost

Chengguo Weng*

*Department of Statistics and Actuarial Science
University of Waterloo, Waterloo, N2L 3G1, Canada.*

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Abstract

Asset management is increasingly integral to the actuarial management of any financial security system. The present paper studies the Constant Proportion Portfolio Insurance (CPPI), one of the popular dynamic strategies of asset management in the pension funds industry. The performance of discrete-time CPPI is studied under proportional trading costs and regime switching. Analytic formulae are developed for a variety of measures used to characterize the risk-and-reward profile of a CPPI portfolio, and a double-sided Laplace inversion method is developed to compute the Omega measure of a CPPI portfolio. The established formulae can be conveniently implemented for sensitivity analysis on performance of a CPPI portfolio. A numerical example with a real data set of S&P 500 index is exploited to illustrate the effects the regime switching feature of the financial market and the existence of transaction costs can exert on the performance of a CPPI portfolio.

Keywords. Portfolio insurance; Markov chain; Gap risk; Regime switching; Shortfall.

*Corresponding author. Email address: c2weng@uwaterloo.ca. Tel: +001(519)888-4567 Ext. 31132.

1 Introduction

Asset management is increasingly integral to the actuarial management of any financial security system due to the development of the equity, derivative and other security markets in the last few decades. Various sophisticated approaches to asset management have been introduced for asset management in the development of actuarial risk management. The present paper focuses on the *Constant Proportion Portfolio Insurance* (CPPI), which is one of the popular dynamic strategies of asset management in the pension funds industry (e.g., Cairns, 2000; Bernard and Kwak, 2014; Graf et al. , 2014).

The CPPI adopts a simplified self-financing strategy and rebalance a portfolio between a *risky asset* (typically a traded fund or index) and a *reserve asset* (typically a bond) dynamically over time. In this portfolio strategy, the portfolio manager starts by setting a *floor* equal to the lowest acceptable value of the portfolio and computes the *cushion* as the excess of the portfolio value over the floor. An amount of capital proportional to the cushion is invested in the risky asset, called the *exposure*, and all the remaining capital are allocated to the reserve asset. The proportional factor is called *multiplier*. The floor and multiplier are exogenously determined according to the investor's risk tolerance. Both the floor and multiplier play a prominent role in the risk-and-reward profile of a CPPI portfolio. This portfolio management strategy implies that, if the cushion value becomes close to zero, the exposure approaches zero too. Therefore, in a continuous-time setting, the management strategy prevents portfolio value from falling below the floor, unless a sharp drop in the investment assets happens before the portfolio manager can modify the portfolio composition.

The CPPI strategy was initially introduced by Perold (1986) for fixed-income instruments and Black and Jones (1987) for equity instruments. It was shown by Perold and Sharpe (1988) that the CPPI strategies implemented in continuous-time trading on assets with prices following geometric Brownian motions are expected HARA utility maximizing (also see Balder and Mahayni, 2010; Branger et. al., 2010). The performance of credit CPPI along with the so-called constant proportion debt obligation structures was studied by Garcia et al. (2008) and Joossens and Schoutens (2010) under a dynamic multivariate jump-driven model for credit spreads. The effect from price jumps on the performance of CPPI strategy was investigated by Cont and Tankov (2009). A general framework of CPPI for investment and protection strategies was formulated by Dersch (2010), and the influence of estimation risk on the performance of CPPI strategies as well as the mitigation effect of the estimation risk by the robustification of mean-variance efficient portfolios were studied by Schöttle and Werner (2010). The effectiveness of the CPPI under a discrete-time setting with transaction costs was studied by Balder et al. (2009) and Balder and Mahayni (2010).

It has been widely recognized and empirically observed that the returns of financial assets are typically non-stationary, and this fact rendered a spur to research on CPPI method with a non-stationary risky asset price process. Weng (2013) recently analyzed the CPPI strategy in a continuous-time setting with the risky asset price modelled by a regime switching exponential Lévy process. Analytic formulae of various risk-and-reward measures for a CPPI portfolio were established, and their implementation was extensively studied for many popular Lévy models including the Merton's jump-diffusion, Kou's jump-diffusion, variance gamma and normal inverse

Gaussian models. More recently, Ameur and Prigent (2014) and Hamidi et al. (2014) proposed a CPPI method with a time-varying multiplier, which is determined, in response to the changes of market conditions, by a quantile and/or expected shortfall based criterion to control the gap risk.

The present paper is a sister paper to Weng (2013) with both in a regime switching setting. The actual economic state usually shows an obvious feature of transition between two or among several states, and the financial return of an asset has distinct characteristics under each economic state. The regime switching framework pioneered by Hamilton (1989) offers a transparent and intuitive way to capture such feature of state transition. In particular, the Markov-modulated regime switching process has been widely advocated in econometrics as well as other relevant areas. For their applications in finance and actuarial science, we refer to Elliott et al. (1995); Hardy (2001); Buffington and Elliott (2002); Elliott et al. (2005); Siu (2005); Li et al. (2008); Shen and Siu (2013) among many others.

Compared to Weng (2013), the present paper has a more realistic model setup. First, the CPPI portfolio in the present paper is rebalanced over a finite set of discrete times, whereas it is continuous-time trading in Weng (2013). Actually, a portfolio can only be revised for a finite number of times over a finite investment horizon. Second, the transaction costs are taken into account in the present paper, but they can hardly be incorporated into a continuous-time model like the one in Weng (2013). In the real market, the transaction cost is typically a prominent factor to consider because it often exerts a substantial effect on the performance of a portfolio management strategy.

In the present paper, the log-returns of both the risky asset and the reserve asset follow distinct distributions at different market states (e.g., bull and bear), and the transition from one market state to another follows a hidden Markov process with a finite state space. A variety of measures, which are defined in subsection 2.3, for the risk-and-reward profile of the CPPI portfolio are established with explicit forms. Moreover, a Laplace inversion methodology is developed for computing the Omega measure of a CPPI portfolio. The results are illustrated with a numerical example with a real financial dataset of S&P 500 index.

The paper proceeds as follows. Section 2 is the model setup, where the regime switching financial market, the discrete-time CPPI strategy, and a set of measures for the risk-and-reward profile of the CPPI portfolio are defined. The analytic formals for various measures are obtained in section 3, and the numerical example is presented in section 4. Section 5 concludes the paper.

2 Model setup

Throughout the paper, all the random elements are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The expectation of a random variable X under the probability measure \mathbb{P} is denoted by $\mathbb{E}[X]$. The transpose of a matrix \mathbf{A} (a vector \mathbf{a}) is denoted by \mathbf{A}' (correspondingly \mathbf{a}'). All the random variables are denoted by capital letters such as X , Y and Z , possibly with a superscript or subscript attached to signify certain meanings accordingly. Capital letters \mathbf{P} and \mathbf{Q} in bold, possibly with a superscript, are exclusively saved to denote matrixes. \mathbf{I} denotes an identity matrix of an appropriate dimension, whereas $\mathbf{1}$ is a column vector with all elements equal to 1. $\mathbb{I}_{\{\cdot\}}$ is the indicator function. $\{H_k, k \geq 0\}$ is used to denote the regime process exclusively.

Greek letters in bold are used to denote a real vector so that $\boldsymbol{\xi}$ denotes (ξ_1, \dots, ξ_r) with real numbers ξ_i , $i = 1, \dots, r$. The real line and the complex plane are respectively denoted by \mathbb{R} and \mathbb{C} . For a vector $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{R}^r$, $\mathcal{D}_{\mathbf{a}}$ denotes the matrix with \mathbf{a} as its diagonal, i.e., $\mathcal{D}_{\mathbf{a}} = \text{diag}(a_1, \dots, a_r)$.

2.1 Regime switching financial market

The CPPI portfolio is allocated between a risky asset $\{S_t, t \geq 0\}$ and a reserve asset $\{B_t, t \geq 0\}$ over a finite set of trading times $\{t_0 < t_1 < \dots < t_n\}$, where $t_0 = 0$ denotes the initial investment time, $t_n = T$ is the terminal trading time, and the other trading times are evenly distributed over the investment horizon $[0, T]$ so that $t_k - t_{k-1} = T/n$ for $k = 1, \dots, n$.

The state of the financial market is described by a discrete-time finite state Markov chain $H = \{H_k, k = 0, 1, \dots, n\}$, where H_k is the market state over period $(t_{k-1}, t_k]$, $k = 1, \dots, n$ and H_0 is the initial market state before entering period $(t_0, t_1]$. In the specific calibration of a regime switching financial market model, two or three states are commonly assumed. They respectively represent a *bullish* and *bearish* state in a model with two states (or regimes), and a third one, if exists, is typically interpreted as an *intermediate* (or *normal*) state between the bullish and bearish states. For presentation convenience, notations from Elliott et al. (1995) will be used in the way that the state space for H consists of r unit vectors $\{h_1, \dots, h_r\}$, where $h_j = (0, \dots, 0, 1, 0, \dots, 0)' \in \mathbb{R}^r$ with 1 in its j th coordinate and 0 in all the others. The convenience of such a representation for the states can be seen from the analysis in the sequel.

For each $j = 1, \dots, r$, let

$$(X^{(j)}, R^{(j)}) = \left\{ \left(X_1^{(j)}, R_1^{(j)} \right), \dots, \left(X_n^{(j)}, R_n^{(j)} \right) \right\} \quad (1)$$

be a sequence of independent and identically distributed bivariate random vectors, where $X_k^{(j)}$ and $R_k^{(j)}$ are, respectively, the log-returns of the risky asset and the reserve asset, given $H_k = h_j$, over period $(t_{k-1}, t_k]$. In other words, $(X_k^{(j)}, R_k^{(j)})$ is the log-return vector of the investment assets over a period of $(t_{k-1}, t_k]$ given that the market stays in the j -th regime. Moreover, $(X^{(j)}, R^{(j)})$ as sequences of random vectors are independent to each other among distinct j 's. The distribution of $(X_k^{(j)}, R_k^{(j)})$ differs from that of $(X_k^{(i)}, R_k^{(i)})$ for $j \neq i$; otherwise, it is not necessary to distinguish the return vector between two regimes i and j .

Put $\mathbf{X}_k = (X_k^{(1)}, \dots, X_k^{(r)})'$ for $k = 1, \dots, n$, and define

$$X_k = \mathbf{X}'_k H_k, \quad k = 1, \dots, n. \quad (2)$$

The resulting sequence $\{X_1, \dots, X_n\}$ are the log-return variables of the risky asset without conditioning on the market state. Similarly, we put $\mathbf{R}_k = (R_k^{(1)}, \dots, R_k^{(r)})'$, $k = 1, \dots, n$. The unconditional log-return sequence $\{R_1, \dots, R_n\}$ of the reserve asset is defined similarly as follows

$$R_k = \mathbf{R}'_k H_k, \quad k = 1, \dots, n. \quad (3)$$

It is worth noting that, although the log-return vectors given in (1) for a fixed regime j are independent and identically distributed, the sequence $\{(X_k, R_k), k = 1, \dots, n\}$ of unconditional log-returns is non-stationary due to the existence of the underlying Markov process H . Moreover,

as commented by Elliott et al. (1995, p16), since H_t takes values in $\{r_1, \dots, r_r\}$, any \mathbb{C}^d -valued function $f(H_t)$ for a positive integer d can be expressed as a linear functional $f(H_t) = \mathbf{f}'H_t$, where $\mathbf{f} = (f(h_1), \dots, f(h_r))'$. Thus,

$$f(H_t) = \sum_{j=1}^r f(h_j)h_j'H_t. \quad (4)$$

This fact will be frequently used in the development of our main results in section 3.

2.2 Discrete-time CPPI under proportional transaction cost

As specified in the preceding subsection, the CPPI portfolio is dynamically rebalanced between a risky asset $\{S_t, t \geq 0\}$ and a reserve asset $\{B_t, t \geq 0\}$ over revision times $t_0 < t_1 < \dots < t_n$, which are evenly distributed over the investment horizon $[0, T]$ so that that $\delta := T/n = t_k - t_{k-1}$, $k = 1, \dots, n$. To distinguish the corresponding quantities between before and after a portfolio revision time t_k , in what follows we shall attach a subscript of t_k^+ on the right lower corner of a notation to signify that the notation represents a quantity immediately after the portfolio revision at time t_k . In contrast, a notation with a subscript t_k means a quantity at time t_k right before the portfolio revision. For example, C_{t_k} and $C_{t_k^+}$, respectively, denote the cushion values before and after the portfolio revision at time t_k , $k = 0, 1, \dots, n$, and similarly, $e_{t_k^+}$ is the exposure in the risky asset immediately after the portfolio revision at time t_k while e_{t_k} is the exposure at time t_k right before the portfolio revision.

2.2.1 Discrete-time CPPI

We suppose that the floor $\{F_t, t \geq 0\}$ is fully determined by the reserve asset price process $\{B_t, t \geq 0\}$ and without loss of any generality, we assume $F_0 = B_0$ and

$$F_t = B_t = F_0 \exp \left\{ \sum_{k=1}^t R_k \right\}, \quad t \geq 0,$$

where F_0 denotes the initial value of the floor and R_k is the log-return variable of the reserve asset over period $(t_{k-1}, t_k]$ as defined in (3). Let V_t denote the CPPI portfolio value at time t so that the cushion $C_t = V_t - F_t$, $t \geq 0$. To avoid the trivial case where the CPPI portfolio value is below the floor at inception of the investment horizon, we assume $V_0 > F_0$.

According to the CPPI strategy, if the cushion $C_{t_k^+} > 0$, the portfolio is revised to include a position of $mC_{t_k^+}$ in the risky asset at time t_k , where $m \geq 1$ is a given constant and called the *multiplier* of the CPPI strategy. The cushion $C_{t_k^+}$ depends on the transaction costs which occur with the portfolio revision at time t_k , in addition to the value of C_{t_k} . To clarify their relationship, we note that the main transaction costs over a financial market, such as a stock market, are typically proportional to the absolute change in exposure. There, we follow Balder et al. (2009) and assume that the transaction cost arising in the portfolio revision at time t_k is given by $\theta |e_{t_k^+} - e_{t_k}|$, where θ is the transaction cost proportional factor, and $|e_{t_k^+} - e_{t_k}|$ is the absolute change in exposure across the revision time t_k . Since the CPPI strategy is self-financing, $\theta |e_{t_k^+} - e_{t_k}|$ is the deduction in portfolio value at time t_k so that $V_{t_k^+} = V_{t_k} - \theta |e_{t_k^+} - e_{t_k}|$ and $C_{t_k^+} = C_{t_k} - \theta |e_{t_k^+} - e_{t_k}|$.

In contrast, if $C_{t_k^+} \leq 0$, the entire portfolio is invested in the reserve asset, which means that the exposure e_{t_k} in risky asset will be sold at time t_k and the transaction cost will be θe_{t_k} , leading to $C_{t_k^+} = C_{t_k} - \theta e_{t_k}$. Combining the analysis for both scenarios yields

$$C_{t_k^+} = \begin{cases} C_{t_k} - \theta |mC_{t_k} - e_{t_k}|, & \text{if } C_{t_k^+} > 0, \\ C_{t_k} - \theta e_{t_k}, & \text{if } C_{t_k^+} \leq 0. \end{cases} \quad (5)$$

Another assumption underlying the relation (5) is that there is no cost associated with the transaction in the reserve asset.

To study the risk and reward profile of the CPPI portfolio, we analyze the evolution of the CPPI portfolio value over the investment time horizon. Note that the CPPI portfolio value is the sum of the floor and the cushion and the floor is fully determined by the price process of the reserve asset. Therefore, it is sufficient for us to focus on the study of the cushion process. For further development, we additionally assume $m\theta < 1$, which is a quite mild condition from a practical point of view, since the transaction cost factor θ is usually as small as 0.3% and a value of 12 is typically viewed as an extremely large scenario for the multiplier m . Consequently, some simple algebras on the system (5) yields

$$C_{t_k^+} = \begin{cases} \frac{C_{t_k} + \theta e_{t_k}}{1 + m\theta}, & \text{if } e_{t_k} \leq mC_{t_k}, \\ \frac{C_{t_k} - \theta e_{t_k}}{1 - m\theta}, & \text{if } m\theta e_{t_k} \leq mC_{t_k} < e_{t_k}, \\ C_{t_k} - \theta e_{t_k}, & \text{if } mC_{t_k} \leq m\theta e_{t_k}. \end{cases} \quad (6)$$

The CPPI strategy revises the exposure to $e_{t_k^+} = \max\{mC_{t_k^+}, 0\}$ at time t_k . Therefore, we get from equation (6) that

$$e_{t_k^+} = \begin{cases} \frac{mC_{t_k} + m\theta e_{t_k}}{1 + m\theta}, & \text{if } e_{t_k} \leq mC_{t_k}, \\ \frac{mC_{t_k} - m\theta e_{t_k}}{1 - m\theta}, & \text{if } m\theta e_{t_k} \leq mC_{t_k} < e_{t_k}, \\ 0, & \text{if } mC_{t_k} \leq m\theta e_{t_k}. \end{cases} \quad (7)$$

It is interesting, but not surprising, to note that, when there is no transaction fee, i.e., $\theta = 0$, the cushion and the exposure (respectively in equations (6) and (7)) become $C_{t_k^+} = C_{t_k}$ and $e_{t_k^+} = \max\{mC_{t_k}, 0\}$, respectively.

To close this subsection, we note that transaction cost occurs immediately at time 0 when the CPPI portfolio is constructed by entering a position of an exposure $e_{t_0^+} = mC_{t_0^+}$ in the risky asset, and it follows from the first case in (6) with $e_{t_0} = 0$ that the initial cushion reduces to $C_{t_0^+} = C_0/(1 + m\theta)$ from C_0 .

2.2.2 Recursion for cushion process

We will develop a recursion between $C_{t_k^+}$ and $C_{t_{k-1}^+}$ to ease the derivation of the terminal cushion value. The expression of (6) for $C_{t_k^+}$ motivates us to investigate how C_{t_k} and e_{t_k} are related to $C_{t_{k-1}^+}$. We shall analyze their relations in two sperate cases. First, when $C_{t_{k-1}^+} \leq 0$, the entire portfolio is invested in the reserve asset from time t_{k-1}^+ on and thus,

$$V_{t_k^+} = \left[F_{t_{k-1}} + C_{t_{k-1}^+} \right] e^{R_k} = F_{t_k} + C_{t_{k-1}^+} e^{R_k},$$

which in turn implies

$$C_{t_k^+} = C_{t_{k-1}^+} e^{R_k} \text{ for } C_{t_{k-1}^+} \leq 0. \quad (8)$$

Second, when $C_{t_{k-1}^+} > 0$, an amount of $mC_{t_{k-1}^+}$ is invested in risky asset and the remainder $V_{t_{k-1}^+} - mC_{t_{k-1}^+} = F_{t_{k-1}^+} - (m-1)C_{t_{k-1}^+}$ is allocated to the reserve asset at time over the period $(t_{k-1}, t_k]$. Recall from the preceding subsection that X_k is the log-return of the risky asset over period $(t_{k-1}, t_k]$, i.e., $X_k = \ln S_{t_k} - \ln S_{t_{k-1}}$, $k = 1, 2, \dots, n$. Therefore, at the moment right before the revision time t_k , the portfolio has an exposure of

$$e_{t_k} = mC_{t_{k-1}^+} e^{X_k} \quad (9)$$

in the risky asset and a dollar amount of

$$\left(F_{t_{k-1}^+} - (m-1)C_{t_{k-1}^+} \right) e^{R_k} = F_{t_k} - (m-1)C_{t_{k-1}^+} e^{R_k}$$

in the reserve asset so that the cushion at time t_k is given by

$$C_{t_k} = mC_{t_{k-1}^+} e^{X_k} - (m-1)C_{t_{k-1}^+} e^{R_k} = e^{R_k} C_{t_{k-1}^+} [m e^{Y_k} - (m-1)], \quad (10)$$

where $Y_k = X_k - R_k$ is the excess return rate of the risky asset over the period $(t_{k-1}, t_k]$, $k = 1, \dots, n$. Substituting (9) and (10) into (6) yields the following recursive formula

$$C_{t_k^+} = C_{t_{k-1}^+} e^{R_k} (\Lambda_k - m\theta \Lambda_k \mathbb{I}_{\{\Lambda_k \leq 0\}}) \text{ for } C_{t_{k-1}^+} > 0, \quad (11)$$

where $\mathbb{I}_{\{\cdot\}}$ is the indicator function, and for $k = 1, \dots, n$,

$$\Lambda_k = \begin{cases} \frac{m(1+\theta)e^{Y_k} - (m-1)}{1+m\theta}, & \text{if } Y_k \geq 0, \\ \frac{m(1-\theta)e^{Y_k} - (m-1)}{1-m\theta}, & \text{if } Y_k < 0. \end{cases} \quad (12)$$

Combining (8) and (11) gives

$$C_{t_k^+} = \begin{cases} C_{t_{k-1}^+} e^{R_k} (\Lambda_k - m\theta \Lambda_k \mathbb{I}_{\{\Lambda_k \leq 0\}}), & \text{if } C_{t_{k-1}^+} > 0, \\ C_{t_{k-1}^+} e^{R_k}, & \text{if } C_{t_{k-1}^+} \leq 0, \end{cases} \quad (13)$$

On the event of $C_{t_{k-1}^+} > 0$, (11) and (12) together imply

$$C_{t_k^+} \leq 0 \text{ if and only if } \Lambda_k \leq 0,$$

and in this case, we say *shortfall* occurs at time t_k . Note that once a shortfall occurs at certain revision time t_k , $V_{t_k^+} \leq F_{t_k^+}$ and the entire portfolio is invested in the reserve asset since then. Therefore, if we define

$$\varrho = \inf \{k \geq 1 : \Lambda_k \leq 0\} = \inf \left\{ k \geq 1 : Y_k \leq \ln \left(\frac{m-1}{m(1-\theta)} \right) \right\}, \quad (14)$$

ϱ is the *shortfall time*. By definition, if $\varrho \leq n$, shortfall occurs within the investment horizon $[0, T]$ and otherwise, no shortfall happens within the investment horizon $[0, T]$. Moreover, for $k = 1, \dots, n$,

$$\{\varrho = k\} = \{\Lambda_j > 0 \ j = 1, \dots, k-1, \text{ and } \Lambda_k \leq 0\},$$

and it follows from (13) that

$$C_{t_k^+} = C_{t_0^+} e^{\sum_{t=1}^k R_t} \prod_{j=1}^{\min(k, \varrho)} (\Lambda_j - m\theta\Lambda_j \mathbb{I}_{\{\Lambda_j \leq 0\}}), \quad k = 1, \dots, n-1. \quad (15)$$

2.2.3 Terminal cushion value

We will develop an expression for the terminal cushion value in this subsection. To this end, we will apply the expression (15) for $C_{t_{n-1}^+}$ and analyze the evolution of the cushion value over the last period $(t_{n-1}, t_n]$ in each of the following two scenarios: (1) $\varrho \leq n-1$ and (2) $\varrho \geq n$. If $\varrho \leq n-1$, the shortfall happens before the last period of $(t_{n-1}, t_n]$ so that all the capital are invested in the reserve asset over the last period. In this case, there will be no transaction cost required to cash the portfolio at the investment terminal time T . Nevertheless, when $\varrho \geq n$, no shortfall happens before the last period, and there is a positive position in the risky asset over the last period. In this case, the portfolio can only be cashed at a value of $V_T - \theta e_T$, where the deduction θe_T is the transaction cost in selling the risky asset. We call the cushion after cashing all the risky asset in the portfolio as the *net terminal cushion*, denoted by C_{T^+} . We shall focus on the net cushion in this paper, as it accurately measures the final payoff of a CPPI portfolio.

We can obtain an explicit expression for the net terminal cushion as shown below. If $\varrho \leq n-1$, (15) reduces to

$$C_{t_{n-1}^+} = C_{t_0^+} e^{\sum_{t=1}^{n-1} R_t} \prod_{j=1}^{\varrho} (\Lambda_j - m\theta\Lambda_j \mathbb{I}_{\{\Lambda_j \leq 0\}}) \leq 0,$$

which indicates that a zero position is in the risky asset and the entire portfolio is invested on the reserve asset over the last period $(t_{n-1}, t_n]$ to earn a log-return rate of R_n . Therefore,

$$\begin{aligned} C_{T^+} &= C_{t_{n-1}^+} e^{R_n} = C_{t_0^+} e^{\sum_{t=1}^n R_t} \prod_{j=1}^{\varrho} (\Lambda_j - m\theta\Lambda_j \mathbb{I}_{\{\Lambda_j \leq 0\}}) \\ &= -C_{t_0^+} e^{\sum_{t=1}^n R_t} \left(\prod_{j=1}^{\varrho-1} \Lambda_j^+ \right) \Xi_{\varrho}^-, \quad \text{if } \varrho \leq n-1, \end{aligned} \quad (16)$$

where

$$\Xi_k = m(1-\theta)e^{Y_k} - (m-1), \quad (17)$$

$\Xi_k^- = \max\{0, -\Xi_k\}$ and $\Xi_k^+ = \max\{0, \Xi_k\}$, $k = 1, \dots, n$.

If $\varrho \geq n$, equation (15) gives

$$C_{t_{n-1}^+} = C_{t_0^+} e^{\sum_{t=1}^{n-1} R_t} \prod_{j=1}^{n-1} \Lambda_j > 0, \quad (18)$$

so that the portfolio value at the end of the last period

$$V_T = mC_{t_{n-1}^+} e^{X_n} + \left[F_{t_{n-1}} - (m-1)C_{t_{n-1}^+} \right] e^{R_n} = F_t + C_{t_{n-1}^+} (me^{X_n} - (m-1)e^{R_n}).$$

In this case the portfolio contains a position of $mC_{t_{n-1}^+} e^{X_n}$ in the risky asset and consequently the liquidation leads to a trading cost of $\theta mC_{t_{n-1}^+} e^{X_n}$ so that the net terminal cushion

$$\begin{aligned}
C_{T^+} &= V_{T^+} - F_T = V_T - \theta mC_{t_{n-1}^+} e^{X_n} - F_T \\
&= C_{t_{n-1}^+} [(m - \theta m)e^{X_n} - (m - 1)e^{R_n}] \\
&= C_{t_0^+} e^{\sum_{t=1}^{n-1} R_t} \left(\prod_{j=1}^{n-1} \Lambda_j \right) e^{R_n} [(m - \theta m)e^{Y_n} - (m - 1)] \\
&= \begin{cases} -C_{t_0^+} e^{\sum_{t=1}^n R_t} \left(\prod_{j=1}^{\varrho-1} \Lambda_j^+ \right) \Xi_{\varrho}^-, & \text{if } \varrho = n, \\ C_{t_0^+} e^{\sum_{t=1}^n R_t} \left(\prod_{j=1}^{n-1} \Lambda_j^+ \right) \Xi_n^+, & \text{if } \varrho > n. \end{cases} \tag{19}
\end{aligned}$$

Finally, we combine (16) and (19) to obtain

$$C_{T^+} = \begin{cases} -C_{t_0^+} e^{\sum_{t=1}^n R_t} \left(\prod_{j=1}^{\varrho-1} \Lambda_j^+ \right) \Xi_{\varrho}^-, & \text{if } \varrho \leq n, \\ C_{t_0^+} e^{\sum_{t=1}^n R_t} \left(\prod_{j=1}^{n-1} \Lambda_j^+ \right) \Xi_n^+, & \text{if } \varrho > n. \end{cases} \tag{20}$$

2.3 Measures for risk-and-reward profile

In this subsection, we will define a set of risk and reward measures for the CPPI portfolio. Developing analytic formulae of these measures is the primary objective of the paper. These measures are motivated by the practical use of the CPPI. While the CPPI strategy can be directly applied by investors, there are CPPI funds: for a CPPI portfolio invested over a finite time investment horizon $[0, T]$ with $T > 0$, the investor pays an initial value of V_0 at time 0 and is guaranteed to receive a value of at least F_T at the terminal time T . If the net terminal portfolio value V_{T^+} is smaller than F_T , a third party will pay the investor the shortfall amount of $F_T - V_{T^+}$. In practice, this guarantee is usually provided by the bank which owns the CPPI portfolio and charges on the investor a premium. As such, at the expiration date T , the investor will receive a payoff of

$$\max\{V_{T^+}, F_T\} = F_T + \max\{C_{T^+}, 0\} = F_T + C_{T^+} \mathbb{I}_{\{C_{T^+} > 0\}},$$

and, in exchange for the premium, the CPPI guarantor will be subject to a gap risk of $C_{T^+} \mathbb{I}_{\{C_{T^+} \leq 0\}}$. Obviously, the evaluation on both quantities are interesting. An insightful investigation on the payoff will be helpful to the investor to develop a comprehensive understanding on his/her risk-and-reward profile in investing a CPPI fund, and a thorough analysis on the gap risk is necessary for the CPPI guarantor to compute a reasonable premium and conduct their own internal risk management.

To evaluate the payoff and the gap risk introduced in the above, we choose the reserve asset as a numeraire and define

$$C_{t_k^+}^* = \frac{C_{t_k^+}}{F_{t_k}}, \text{ where } F_{t_k} = F_0 \exp\left(\sum_{t=1}^k R_t\right), \quad = 0, 1, \dots, n.$$

From (20),

$$C_{T^+}^* = \begin{cases} -C_{t_0^+}^* \left(\prod_{j=1}^{\varrho-1} \Lambda_j^+ \right) \Xi_{\varrho}^-, & \text{if } \varrho \leq n, \\ C_{t_0^+}^* \left(\prod_{j=1}^{n-1} \Lambda_j^+ \right) \Xi_n^+, & \text{if } \varrho > n, \end{cases} \tag{21}$$

where Λ_j and Ξ_j are given by (12) and (17), respectively.

We evaluate the payoff for the investor and the gap risk for the guarantor at time 0 by using the reserve asset price process for discounting, so that the time 0 value of the payoff is

$$1 + \max \left\{ \frac{C_{T+}}{F_T}, 0 \right\} = 1 + \max \{ C_{T+}^*, 0 \} \quad (22)$$

and the time 0 value of the gap risk is $C_{T+}^* \mathbb{I}_{\{C_{T+}^* \leq 0\}}$. Note that the three events $\{\varrho \leq n\}$, $\{C_T \leq 0\}$ and $\{C_T^* \leq 0\}$ are eventually the same, all indicating that a gap risk comes up during the investment time horizon $[0, T]$. Therefore, the time 0 value of the gap risk can be equivalently expressed by

$$C_{T+}^* \mathbb{I}_{\{C_{T+}^* \leq 0\}} = C_{T+}^* \mathbb{I}_{\{\varrho \leq n\}}. \quad (23)$$

Upon the expressions in (22) and (23), it is interesting to investigate the following quantities to characterize the risk-and-reward profile of a CPPI portfolio:

- i) *Shortfall probability* defined by $\text{SP}_{H_0} = \Pr(C_{T+}^* < 0 | H_0)$,
- ii) *Unconditional expected shortfall* defined by $\text{UES}_{H_0} = \mathbb{E} \left(-C_{T+}^* \mathbb{I}_{\{C_{T+}^* \leq 0\}} \middle| H_0 \right)$,
- iii) *Conditional expected shortfall* defined by $\text{CES}_{H_0} = \mathbb{E}(-C_{T+}^* | C_{T+}^* \leq 0, H_0)$,
- iv) *Unconditional expected gain* defined by $\text{UEG}_{H_0} = \mathbb{E} \left(C_{T+}^* \mathbb{I}_{\{C_{T+}^* > 0\}}, H_0 \right)$,
- v) *Conditional expected gain* defined by $\text{CEG}(T) = \mathbb{E}(C_{T+}^* | C_{T+}^* > 0, H_0)$,

where the presence of H_0 in each measure signifies the dependence of these measures on the initial regime state of the market. Additionally, we will study the *Omega measure* (see subsection 3.5 for the precise definition) of the discounted terminal portfolio value V_{T+}^* by a Laplace transform methodology.

3 Analytic formulae of the risk and reward measures

3.1 Preliminaries on regime switching model

To ease the presentation of further development, define for each $t \geq 1$ and $j = 1, \dots, r$,

$$\Gamma_j(t) = \sum_{k=1}^t h'_j H_k, \quad (24)$$

which is the total number of periods over which the regime process H spends in state h_j over the time horizon $(0, t]$. Clearly, $\sum_{j=1}^r \Gamma_j(t) = t$ almost surely, $t \geq 0$, and moreover, for any vector of complex numbers $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{C}^r$,

$$\sum_{j=1}^r a_j \Gamma_j(t) = \sum_{k=1}^t \mathbf{a}' H_k. \quad (25)$$

Let $\mathbf{P} = (p_{ij})$ be the one-step transition probability matrix of the Markov process H such that $p_{ij} = \Pr(\rho_t = h_i | \rho_{t-1} = h_j)$, $t \geq 1$, $i, j = 1, \dots, h$.

The two lemmas below will be used for the development of our main results in the subsequent sections. The proof of (27) has been given by Elliott et al. (1995, p.17). The result (26) is developed on its own interest.

Lemma 3.1. *The regime process $\{H_t, t \geq 0\}$ admits the following representation*

$$H_t = H_0 + \sum_{i=0}^{t-1} (\mathbf{P} - \mathbf{I})H_i + M_t, \quad t \geq 1, \quad (26)$$

so that

$$H_t = \mathbf{P}H_{t-1} + A_t, \quad t \geq 1, \quad (27)$$

where $A_t = M_t - M_{t-1}$ and $\{M_t, t \geq 0\}$ with $M_0 = 0$ is a martingale with respect to $\{\mathcal{F}_t^H, t \geq 0\}$, the natural filtration generated by process $\{H_t, t \geq 0\}$.

Proof. Let $M_t = H_t - H_0 - \sum_{i=0}^{t-1} (\mathbf{P} - \mathbf{I})H_i$, $t \geq 1$. It is sufficient to show that $\{M_t, t \geq 0\}$ is a martingale with respect to $\{\mathcal{F}_t^H, t \geq 0\}$. Indeed, $M_1 = H_1 - \mathbf{P}H_0$ holds trivially and thus, $\mathbb{E}[M_1 | \mathcal{F}_0^H] = \mathbb{E}[H_1 - \mathbf{P}H_0 | H_0] = 0$. For any $t \geq 2$,

$$\begin{aligned} \mathbb{E}[M_t | \mathcal{F}_{t-1}^H] &= \mathbb{E} \left[\left(H_t - H_0 - \sum_{i=0}^{t-1} (\mathbf{P} - \mathbf{I})H_i \right) \middle| \mathcal{F}_{t-1}^H \right] \\ &= \mathbb{E}[H_t | \mathcal{F}_{t-1}^H] - H_0 - \sum_{i=0}^{t-1} (\mathbf{P} - \mathbf{I})H_i \\ &= \mathbf{P}H_{t-1} - H_0 - \sum_{i=0}^{t-1} (\mathbf{P} - \mathbf{I})H_i \\ &= H_{t-1} - H_0 - \sum_{i=0}^{t-2} (\mathbf{P} - \mathbf{I})H_i \\ &= M_{t-1}, \end{aligned}$$

by which the proof is complete. \square

Lemma 3.2. *Denote $\mathcal{D}_{\mathbf{a}} = \text{diag}(e^{a_1}, \dots, e^{a_r})$ for a vector $\mathbf{a} = (a_1, \dots, a_r)' \in \mathbb{C}^r$. Then, for any integer $t \geq 1$,*

$$\begin{aligned} (a) \quad &\mathbb{E} \left[\exp \left(\sum_{j=1}^r a_j \Gamma_j(t) \right) H_{t+k} \middle| H_0 \right] = \mathbf{P}^k (\mathcal{D}_{\mathbf{a}} \mathbf{P})^t H_0, \quad k = 0, 1, 2, \dots, \text{ and} \\ (b) \quad &\mathbb{E} \left[\exp \left(\sum_{j=1}^r a_j \Gamma_j(t) \right) \middle| H_0 \right] = \mathbf{1}' (\mathcal{D}_{\mathbf{a}} \mathbf{P})^t H_0. \end{aligned}$$

Proof. (a) By the Markov property of H and the linear representation (4), we obtain

$$\begin{aligned} \mathbb{E} \left[e^{\mathbf{a}' H_t} H_t | \mathcal{F}_{t-1}^H \right] &= \mathbb{E} \left[\sum_{j=1}^r e^{\mathbf{a}' h_j} h_j h_j' H_t \middle| \mathcal{F}_{t-1}^H \right] = \mathbb{E} \left[\text{diag}(e^{a_1}, \dots, e^{a_r}) H_t | \mathcal{F}_{t-1}^H \right] \\ &= \mathcal{D}_{\mathbf{a}} \mathbf{P} H_{t-1}, \end{aligned} \quad (28)$$

where the last step is due to Lemma 3.1. To proceed, let $U_t = \exp\left(\sum_{j=1}^r a_j \Gamma_j(t)\right)$ and $\mathbf{V}_t = U_t H_t$. Then, (28) implies

$$\mathbb{E}[\mathbf{V}_1 | H_0] = \mathbb{E}\left[e^{\sum_{j=1}^r a_j \Gamma_j(1)} H_1 \middle| H_0\right] = \mathbb{E}\left[e^{\mathbf{a}' H_1} H_1 \middle| H_0\right] = \mathcal{D}_{\mathbf{a}} \mathbf{P} H_0, \quad (29)$$

where the last equality follows in the same way as (28). For $t \geq 2$, we apply (25) to obtain $U_t = U_{t-1} e^{\mathbf{a}' H_t}$, whereby

$$\begin{aligned} \mathbb{E}[\mathbf{V}_t | \mathcal{F}_{t-1}^H] &= \mathbb{E}[U_t H_t | \mathcal{F}_{t-1}^H] = \mathbb{E}[U_{t-1} e^{\mathbf{a}' H_t} H_t | \mathcal{F}_{t-1}^H] = U_{t-1} \mathbb{E}[e^{\mathbf{a}' H_t} H_t | \mathcal{F}_{t-1}^H] \\ &= U_{t-1} \mathcal{D}_{\mathbf{a}} \mathbf{P} H_{t-1} = \mathcal{D}_{\mathbf{a}} \mathbf{P} \mathbf{V}_{t-1}, \end{aligned}$$

which implies $\mathbb{E}[\mathbf{V}_t | H_0] = \mathcal{D}_{\mathbf{a}} \mathbf{P} \mathbb{E}[\mathbf{V}_{t-1} | H_0]$. Combining this with (29) yields $\mathbb{E}[\mathbf{V}_t | H_0] = (\mathcal{D}_{\mathbf{a}} \mathbf{P})^t H_0$. This means that the desired result holds for $k = 0$.

By Lemma 3.1, $H_{t+1} = \mathbf{P} H_t + A_{t+1}$, and therefore,

$$\begin{aligned} \mathbb{E}\left(e^{\sum_{j=1}^r a_j \Gamma_j(t)} H_{t+1} \middle| H_0\right) &= \mathbb{E}\left[\mathbf{P} \cdot e^{\sum_{j=1}^r a_j \Gamma_j(t)} H_t \middle| H_0\right] + \mathbb{E}\left[\mathbb{E}\left(e^{\sum_{j=1}^r a_j \Gamma_j(t)} A_{t+1} \middle| \mathcal{F}_t^H\right) \middle| H_0\right] \\ &= \mathbb{E}[\mathbf{P} \mathbf{V}_t | H_0] + \mathbb{E}\left[e^{\sum_{j=1}^r a_j \Gamma_j(t)} \mathbb{E}(A_{t+1} | \mathcal{F}_t^H) \middle| H_0\right] \\ &= \mathbf{P} \mathbb{E}[\mathbf{V}_t | H_0] \\ &= \mathbf{P} (\mathcal{D}_{\mathbf{a}} \mathbf{P})^t H_0, \end{aligned}$$

which proves the desired result for $k = 1$. The result for a general k can be obtained by induction.

(b) From definition of \mathbf{V}_t , $\exp\left(\sum_{j=1}^r a_j \Gamma_j(t)\right) = \mathbf{1}' \mathbf{V}_t$. Taking expectation on both sides of the equation and applying the result of part (a) leads to the desired result. \square

3.2 Shortfall probability

Recall from (12) that the excess return rate of the risky asset is defined by $Y_k = X_k - R_k$, where X_k and R_k are given in (2) and (3), respectively. Thus, if we put $\mathbf{Y}_k = (Y_k^{(1)}, \dots, Y_k^{(r)})$ with $Y_k^{(j)} = X_k^{(j)} - R_k^{(j)}$ for $k = 1, \dots, n$ and $j = 1, \dots, r$, then the excess return rate $Y_k = \mathbf{Y}_k' H_k$, $k = 1, \dots, n$, and $\{Y_1, \dots, Y_n\}$ are conditionally independent given (H_1, \dots, H_n) . We further denote

$$\mathbf{s} = (s_1, \dots, s_r)', \quad \text{where } s_j = \ln \left[\Pr \left(Y_t^{(j)} > \ln \left(\frac{m-1}{m(1-\theta)} \right) \right) \right], \quad j = 1, \dots, r. \quad (30)$$

Proposition 3.1. *The shortfall probability for the CPPI portfolio is given by*

$$\text{SP}_{H_0} = \Pr(\rho \leq n | H_0) = \Pr(\mathbf{C}_{T^+}^* \leq 0 | H_0) = 1 - \mathbf{1}' (\mathcal{D}_{\mathbf{s}} \mathbf{P})^n H_0, \quad (31)$$

where $\mathcal{D}_{\mathbf{s}} = \text{diag}(e^{s_1}, \dots, e^{s_r})$.

Proof. By the definition of ϱ in (14), $\{\rho > n\} = \{Y_k > \ln\left(\frac{m-1}{m(1-\theta)}\right), k = 1, \dots, n\}$ and therefore,

it follows from the conditional independence of $\{Y_1, \dots, Y_n\}$ that

$$\begin{aligned}
\text{SP}_{H_0} &= 1 - \mathbb{P} \left(Y_k > \ln \left(\frac{m-1}{m(1-\theta)} \right), k = 1, \dots, n \middle| H_0 \right) \\
&= 1 - \mathbb{E} \left[\Pr \left(Y_k > \ln \left(\frac{m-1}{m(1-\theta)} \right), k = 1, \dots, n \middle| H_1, \dots, H_n \right) \middle| H_0 \right] \\
&= 1 - \mathbb{E} \left[\prod_{k=1}^n \Pr \left(Y_k > \ln \left(\frac{m-1}{m(1-\theta)} \right) \middle| H_k \right) \middle| H_0 \right] \\
&= 1 - \mathbb{E} \left[\prod_{k=1}^n (e^{\mathbf{s}})' H_k \middle| H_0 \right] = 1 - \mathbb{E} \left[\exp \left(\sum_{k=1}^n \mathbf{s}' H_k \right) \middle| H_0 \right],
\end{aligned}$$

where $e^{\mathbf{s}}$ denotes the vector $(e^{s_1}, \dots, e^{s_r})$. Consequently, the desired result follows from part (b) of Lemma 3.2 as follows

$$\begin{aligned}
\text{SP}_{H_0} &= 1 - \mathbb{E} \left[\exp \left(\sum_{k=1}^n \mathbf{s}' H_k \right) \middle| H_0 \right] = 1 - \mathbb{E} \left[\exp \left(\sum_{j=1}^r s_j \Gamma_j(n) \right) \middle| H_0 \right] \\
&= 1 - \mathbf{1}' (\mathcal{D}_{\mathbf{s}} \mathbf{P})^n H_0.
\end{aligned}$$

□

Remark 3.1. (a) Obviously, $\ln \left(\frac{m-1}{m(1-\theta)} \right)$ increases with m so that s_j defined in (30) decreases with m . This, along with (31), further implies that the shortfall probability increases with the multiplier m . This observation accords to our intuition well on a CPPI portfolio: the higher the multiplier, the more capital invested in the risky asset and thus the faster the portfolio approaches the floor when there is a sustained decrease in the risky asset price.

(b) From Proposition 3.1, we can determine an upper bound on the multiplier m for the CPPI portfolio to satisfy a quantile condition, following the same idea as in Ameur and Prigent (2014). More specifically, to control the shortfall probability SP_{H_0} with certain tolerance level ϵ , one needs to choose a multiplier to satisfy $\text{SP}_{H_0} \leq \epsilon$, which, by Proposition 3.1 and the fact that $\mathbf{1}' (\mathcal{D}_{\mathbf{s}} \mathbf{P})^n H_0$ is decreasing in the multiplier m , is equivalent to choosing a multiplier no larger than a value for m which satisfies

$$\mathbf{1}' (\mathcal{D}_{\mathbf{s}} \mathbf{P})^n H_0 = 1 - \epsilon. \tag{32}$$

(c) The proof of Proposition 3.1 essentially provides us with a way to compute the distribution of the minimum statistics $Y_{\min} = \min\{Y_k, k = 1, \dots, n\}$. Let $s_j(x) = \ln \left[\Pr \left(Y_t^{(j)} > x \right) \right]$, $x \in \mathbb{R}$, $j = 1, \dots, r$, by a little abuse of notation. Then, the same lines with $\ln \left(\frac{m-1}{m(1-\theta)} \right)$ replaced by x in the proof of Proposition 3.1 yields

$$F_{Y_{\min}}(x|H_0) := \Pr(Y_{\min} \leq x | H_0) = 1 - \mathbf{1}' (\mathcal{D}_{\mathbf{s}(x)} \mathbf{P})^n H_0, \quad x \in \mathbb{R},$$

where $\mathbf{s}(x) = (s_1(x), \dots, s_r(x))'$ and $\mathcal{D}_{\mathbf{s}(x)} = \text{diag}(e^{s_1(x)}, \dots, e^{s_r(x)})$. Since

$$\text{SP}_{H_0} = F_{Y_{\min}} \left(\ln \left(\frac{m-1}{m(1-\theta)} \right) \middle| H_0 \right),$$

the quantile condition $\text{SP}_{H_0} \leq \epsilon$ mentioned in part (b) for multiplier m is equivalent to

$$m \leq \frac{1}{1 - (1 - \theta) \exp\left(F_{Y_{\min}}^{-1}(\epsilon|H_0)\right)},$$

where $F_{Y_{\min}}^{-1}(\cdot|H_0)$ denotes the inverse of $F_{Y_{\min}}(\cdot|H_0)$.

3.3 Expected shortfall

Recall the definition of random variables Λ_k in (12). We similarly define $\Lambda_k^{(j)}$ by replacing Y_k with $Y_k^{(j)}$ in (12) for $k = 1, \dots, n$ and $j = 1, \dots, r$. We use $\Lambda_k^{(j)+}$ and $\Lambda_k^{(j)-}$ to denote their positive and negative parts, respectively. We similarly define $\Xi_k^{(j)}$ and their positive part $\Xi_k^{(j)+}$ and negative part $\Xi_k^{(j)-}$ by replacing Y_k with $Y_k^{(j)}$ in (17). Note that $\{(Y_k^{(j)}, \Lambda_k^{(j)}, \Xi_k^{(j)}), k = 1, \dots, n\}$ as a sequence of random vectors are independent and identically distributed for a fixed $j = 1, \dots, r$. Furthermore, we denote $\boldsymbol{\lambda}^+ = (\lambda_1^+, \dots, \lambda_r^+)'$ and $\boldsymbol{\xi}^- = (\xi_1^-, \dots, \xi_r^-)'$ with

$$\lambda_j^+ = \ln \mathbb{E} \left[\Lambda_k^{(j)+} \right] \quad \text{and} \quad \xi_j^- = \mathbb{E} \left[\Xi_k^{(j)-} \right], \quad j = 1, \dots, r.$$

Proposition 3.2. *The unconditional expected shortfall of the CPPI portfolio is given by*

$$\begin{aligned} \text{UES}_{H_0} &= \mathbb{E} \left[-C_{T^+}^* \mathbb{I}_{\{\rho \leq n\}} \right] = \mathbb{E} \left[-C_{T^+}^* \mathbb{I}_{\{C_{T^+}^* \leq 0\}} \right] \\ &= C_{t_0^+}^* \cdot (\boldsymbol{\xi}^-)' \left(\mathbf{I} - \mathbf{P} \cdot \mathcal{D}_{\boldsymbol{\lambda}^+} \right)^{-1} \mathbf{P} \left(\mathbf{I} - \left(\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P} \right)^n \right) H_0, \end{aligned} \quad (33)$$

provided that $(\mathbf{I} - \mathbf{P} \cdot \mathcal{D}_{\boldsymbol{\lambda}^+})$ is invertible, where $\mathcal{D}_{\boldsymbol{\lambda}^+} = \text{diag}(e^{\lambda_1^+}, \dots, e^{\lambda_r^+})$.

Proof. Let $B_k = -C_{T^+}^* \mathbb{I}_{\{\rho=k\}}$ for $k = 1, \dots, n$. Then, by (21),

$$B_k = C_{t_0^+}^* \left(\prod_{t=1}^{k-1} \Lambda_t^+ \right) \cdot \Xi_k^-, \quad k = 1, \dots, n. \quad (34)$$

Further define

$$J_k^{(j)} = \prod_{t=1}^k \left(\Lambda_t^{(j)+} \right) = \exp \left(\sum_{t=1}^k \ln \Lambda_t^{(j)+} \right), \quad j = 1, \dots, r \quad \text{and} \quad k = 1, \dots, n.$$

Since $\{\Lambda_t^{(j)+}, t = 1, \dots, n\}$ are independent and identically distributed for a fixed j , $\ln J_k^{(j)}$ has independent and identically distributed increments as a stochastic process indexed by k .

For $k = 1$, trivially we have

$$\mathbb{E}[B_1|H_0] = C_{t_0^+}^* \mathbb{E}[\Xi_k^-|H_0] = C_{t_0^+}^* \cdot (\boldsymbol{\xi}^-)' \mathbf{P} H_0. \quad (35)$$

Next, we consider $k \geq 2$, and let M be the number of transitions which the regime switching process H experiences over the time horizon $[0, k-1)$. Note that the regime transition over $[0, k-1)$ can only occur at times t_0, t_1, \dots, t_{k-2} . Denote $\tau_0 = 0$ and $\tau_{M+1} = k-1$, and let $\tau_i = l$ if the i th transition occurs at time t_l for $l = 0, 1, \dots, k-2$ and $i = 1, \dots, M$. Note that τ_1 can be 0, which means that the first transition occurs at time 0 so that $H_1 \neq H_0$. Further let η_j

denote the corresponding inter-transition times such that $\eta_j = \tau_j - \tau_{j-1}$ for $j = 1, \dots, M$ and write $\eta_{M+1} = \tau_{M+1} - \tau_M$. Let $h_{K_{j+1}}$ denote the state in which the process H stays over period $(\tau_j, \tau_{j+1}]$, i.e., H stays in the K_{j+1} -th state over period $(\tau_j, \tau_{j+1}]$, $j = 0, 1, \dots, M$. Then,

$$\prod_{j=1}^{k-1} \Lambda_j^+ = \prod_{j=0}^M \left(\frac{J_{\tau_{j+1}}^{(K_{j+1})}}{J_{\tau_j}^{(K_{j+1})}} \right), \quad k \geq 2.$$

Noticing that $\{(\Lambda_1, \Xi_1), \dots, (\Lambda_k, \Xi_k)\}$ are conditionally independent given $\mathcal{G}_k^H := \sigma\{H_1, \dots, H_k\}$, we obtain

$$\begin{aligned} \mathbb{E} \left[\left(\prod_{j=1}^{k-1} \Lambda_j^+ \right) \middle| \mathcal{G}_k^H \right] &= \mathbb{E} \left[\prod_{j=0}^M \left(\frac{J_{\tau_{j+1}}^{(K_{j+1})}}{J_{\tau_j}^{(K_{j+1})}} \right) \middle| \mathcal{G}_k^H \right] = \prod_{j=0}^M \mathbb{E} \left[J_{\eta_{j+1}}^{(K_{j+1})} \middle| K_{j+1} \right] \\ &= \prod_{j=0}^M \left[\mathbb{E} \left(\Lambda_1^{(K_{j+1})+} \middle| K_{j+1} \right) \right]^{\eta_{j+1}} = \prod_{j=0}^M e^{\eta_{j+1} \lambda_{K_{j+1}}^+} \\ &= \exp \left(\sum_{j=0}^M \eta_{j+1} \lambda_{K_{j+1}}^+ \right), \end{aligned}$$

and hence it follows from (34) that

$$\begin{aligned} \mathbb{E} [B_k | \mathcal{G}_k^H] &= C_{t_0^+}^* \cdot \mathbb{E} \left[\left(\prod_{j=1}^{k-1} \Lambda_j^+ \right) \middle| \mathcal{G}_k^H \right] \cdot \mathbb{E} (\Xi_k^- | \mathcal{G}_k^H) \\ &= C_{t_0^+}^* \cdot \exp \left(\sum_{j=0}^M \eta_{j+1} \lambda_{K_{j+1}}^+ \right) \cdot (\boldsymbol{\xi}^-)' H_k. \end{aligned}$$

Note that $\sum_{j=0}^M \eta_{j+1} \lambda_{K_{j+1}}^+ = \sum_{j=1}^r (\lambda_j^+ \Gamma_j(k-1))$, where $\Gamma_j(k-1)$ is defined in (24). Therefore, by part (a) of Lemma 3.2, we obtain

$$\begin{aligned} \mathbb{E} [B_k | H_0] &= \mathbb{E} \left[C_{t_0^+}^* \cdot \exp \left(\sum_{j=1}^r (\Gamma_j(k-1) \cdot \lambda_j^+) \right) \cdot (\boldsymbol{\xi}^-)' H_k \middle| H_0 \right] \\ &= C_{t_0^+}^* \cdot (\boldsymbol{\xi}^-)' \mathbb{E} \left[\exp \left(\sum_{j=1}^r (\lambda_j^+ \Gamma_j(k-1)) \right) \cdot H_k \middle| H_0 \right] \\ &= C_{t_0^+}^* \cdot (\boldsymbol{\xi}^-)' \mathbf{P} (\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P})^{k-1} H_0, \quad k \geq 2. \end{aligned} \tag{36}$$

Combining (35) and (36), we obtain the unconditional expected shortfall as follows

$$\begin{aligned} \text{UES}_{H_0} &= \mathbb{E} [-C_{T^+}^* \mathbb{I}_{\{\varrho \leq n\}} | H_0] = \sum_{k=1}^n \mathbb{E} [-C_{T^+}^* \mathbb{I}_{\{\varrho=k\}} | H_0] = \sum_{k=1}^n \mathbb{E} [B_k] \\ &= C_{t_0^+}^* \cdot \sum_{k=1}^n (\boldsymbol{\xi}^-)' \mathbf{P} (\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P})^{k-1} H_0 \\ &= C_{t_0^+}^* \cdot (\boldsymbol{\xi}^-)' (\mathbf{I} - \mathbf{P} \cdot \mathcal{D}_{\boldsymbol{\lambda}^+})^{-1} \mathbf{P} (\mathbf{I} - (\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P})^n) H_0, \end{aligned}$$

provided that $(\mathbf{I} - \mathbf{P} \cdot \mathcal{D}_{\boldsymbol{\lambda}^+})$ is invertible. \square

Remark 3.2. Combining Propositions 3.1 and 3.2 yields the following formula for the conditional expected shortfall:

$$\begin{aligned} \mathbb{E}[-C_{T+}^* | \varrho \leq n, H_0] &\equiv \mathbb{E}[-C_{T+}^* | C_{T+}^* \leq 0, H_0] \\ &= \frac{C_{t_0^+}^* (\boldsymbol{\xi}^-)' (\mathbf{I} - \mathbf{P} \cdot \mathcal{D}_{\boldsymbol{\lambda}^+})^{-1} \mathbf{P} (\mathbf{I} - (\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P})^n) H_0}{1 - \mathbf{1}' (\mathcal{D}_{\mathbf{s}} \mathbf{P})^n H_0}. \end{aligned}$$

Moreover, the proof of Proposition 3.2 also implies that the ν -th moment of the shortfall of the CPPI portfolio, if exists, can be computed by the following formula

$$\begin{aligned} &\mathbb{E} [(-C_{T+}^*)^\nu \mathbb{I}_{\{\varrho \leq n\}}] \\ &= \mathbb{E} [(-C_{T+}^*)^\nu \mathbb{I}_{\{C_{T+}^* \leq 0\}}] \\ &= (C_{t_0^+}^*)^\nu \cdot (\boldsymbol{\xi}(\nu)^-)' (\mathbf{I} - \mathbf{P} \cdot \mathcal{D}_{\boldsymbol{\lambda}(\nu)^+})^{-1} \mathbf{P} (\mathbf{I} - (\mathcal{D}_{\boldsymbol{\lambda}(\nu)^+} \mathbf{P})^n) H_0, \end{aligned}$$

provided that $(\mathbf{I} - \mathbf{P} \cdot \mathcal{D}_{\boldsymbol{\lambda}(\nu)^+})$ is invertible, where

$$\mathcal{D}_{\boldsymbol{\lambda}(\nu)^+} = \text{diag} \left(\mathbb{E} \left[(\Lambda_1^{(j)+})^\nu \right], \dots, \mathbb{E} \left[(\Lambda_k^{(j)+})^\nu \right] \right) \quad (37)$$

$$\boldsymbol{\lambda}(\nu)^+ = \left(\ln \mathbb{E} \left[(\Lambda_1^{(j)+})^\nu \right], \dots, \ln \mathbb{E} \left[(\Lambda_k^{(j)+})^\nu \right] \right) \quad (38)$$

$$\boldsymbol{\xi}(\nu)^- = \left(\ln \mathbb{E} \left[(\Xi_1^{(j)-})^\nu \right], \dots, \ln \mathbb{E} \left[(\Xi_k^{(j)-})^\nu \right] \right) \quad (39)$$

3.4 Expected gain

An analytic expression for the (unconditional and conditional) expected gain can be obtained in a similar way as what have been done for the expected shortfall in the preceding subsection. Define

$$\boldsymbol{\xi}^+ = (\xi_1^+, \dots, \xi_r^+)', \text{ where } \xi_j^+ = \mathbb{E} \left[\Xi_1^{(j)+} \right], \quad j = 1, \dots, r.$$

Proposition 3.3. The unconditional expected gain is given by

$$\begin{aligned} \text{UEG}_{H_0} &= \mathbb{E} [C_{T+}^* \mathbb{I}_{\{\varrho > n\}} | H_0] = \mathbb{E} [C_{T+}^* \mathbb{I}_{\{C_{T+}^* > 0\}} | H_0] \\ &= C_{t_0^+}^* \cdot (\boldsymbol{\xi}^+)' \mathbf{P} (\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P})^{n-1} H_0, \end{aligned} \quad (40)$$

where $\mathcal{D}_{\boldsymbol{\lambda}^+}$ is a diagonal matrix defined in Proposition 3.2.

Proof. From (21), $C_{T+}^* = C_{t_0^+}^* \left(\prod_{j=1}^{n-1} \Lambda_j^+ \right) \cdot \Xi_n^+$ for $\varrho > n$. Similar to (36), it follows from part (b) of Lemma 3.2 that

$$\begin{aligned} \mathbb{E} [C_{T+}^* \mathbb{I}_{\{\varrho > n\}} | H_0] &= \mathbb{E} \left[C_{t_0^+}^* \cdot \exp \left(\sum_{j=1}^r (\Gamma_j(n-1) \cdot \lambda_j^+) \right) \cdot (\boldsymbol{\xi}^+)' H_n \middle| H_0 \right] \\ &= C_{t_0^+}^* \cdot (\boldsymbol{\xi}^+)' \mathbb{E} \left[\exp \left(\sum_{j=1}^r (\Gamma_j(n-1) \cdot \lambda_j^+) \right) \cdot H_n \middle| H_0 \right] \\ &= C_{t_0^+}^* \cdot (\boldsymbol{\xi}^+)' \mathbf{P} (\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P})^{n-1} H_0, \end{aligned}$$

by which the proof is complete. \square

Remark 3.3. *The conditional expected gain can be computed by using Propositions 3.1 and 3.3 as follows*

$$\begin{aligned}\mathbb{E}[C_{T+}^* | \varrho > n, H_0] &= \mathbb{E}[C_{T+}^* | C_{T+}^* > 0, H_0] = \frac{\mathbb{E}[C_{T+}^* \mathbb{I}_{\{C_{T+}^* > 0\}} | H_0]}{\Pr(C_{T+}^* > 0 | H_0)} \\ &= C_{t_0^+}^* \cdot \frac{(\boldsymbol{\xi}^+)' \mathbf{P} (\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P})^{n-1} H_0}{\mathbf{1}' (\mathbf{P}_s)^n \cdot \mathbb{E}[H_0]}.\end{aligned}$$

Moreover, by combining Propositions 3.2 and 3.3, the expected terminal cushion can be obtained as follows

$$\begin{aligned}\mathbb{E}[C_{T+}^* | H_0] &= \mathbb{E}[C_{T+}^* \mathbb{I}_{\{\varrho > n\}} | H_0] + \mathbb{E}[C_{T+}^* \mathbb{I}_{\{\varrho \leq n\}} | H_0] \\ &= C_{t_0^+}^* \cdot (\boldsymbol{\xi}^+)' \mathbf{P} (\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P})^{n-1} H_0 \\ &\quad - C_{t_0^+}^* \cdot (\boldsymbol{\xi}^-)' (\mathbf{I} - \mathbf{P} \cdot \mathcal{D}_{\boldsymbol{\lambda}^+})^{-1} \mathbf{P} (\mathbf{I} - (\mathcal{D}_{\boldsymbol{\lambda}^+} \mathbf{P})^n) H_0,\end{aligned}$$

provided that $(\mathbf{I} - \mathbf{P} \cdot \mathcal{D}_{\boldsymbol{\lambda}^+})$ is invertible. Finally, the ν -th moment of the shortfall can be calculated as follows

$$\begin{aligned}\mathbb{E}[(C_{T+}^*)^\nu \mathbb{I}_{\{\varrho > n\}} | H_0] &= \mathbb{E}[(C_{T+}^*)^\nu \mathbb{I}_{\{C_{T+}^* > 0\}} | H_0] \\ &= (C_{t_0^+}^*)^\nu \cdot (\boldsymbol{\xi}(\nu)^+)' \mathbf{P} (\mathcal{D}_{\boldsymbol{\lambda}(\nu)^+} \mathbf{P})^{n-1} H_0,\end{aligned}$$

where $\boldsymbol{\lambda}(\nu)^+$ is a vector defined in (41) and

$$\boldsymbol{\xi}(\nu)^+ = \left(\ln \mathbb{E} \left[\left(\Xi_1^{(j)+} \right)^\nu \right], \dots, \ln \mathbb{E} \left[\left(\Xi_k^{(j)+} \right)^\nu \right] \right).$$

3.5 Omega measure

The payoffs of portfolio insurance strategies are typically non-linear with respect to the risky reference asset, which induces asymmetric return distributions. Traditional performance measures such as mean-variance, mean-lower partial moment (for instance, the mean-CVaR), and Sharpe's ratio are believed inadequate to evaluate the performance of a CPPI portfolio. Bertrand and Prigent (2011) introduced Kappa performance measures and especially the Omega measure to take account of the entire return distribution of the CPPI terminal portfolio value. It is designed to overcome the shortcomings of performance measures based only on the mean and certain particular (partial) moment of the distribution of the terminal portfolio value.

The Omega measure was first introduced by Keating and Shadwick (2002); Cascon, et al. (2003). This measure splits the return into two sub-parts according to a threshold which corresponds to a minimum acceptable return. More precisely, the Omega measure for a profit-loss random variable Z is defined as the probability weighted ratio of gains to losses relative to a return threshold L as follows

$$\Omega(Z; L) = \frac{\mathbb{E}[(Z - L)_+]}{\mathbb{E}[(L - Z)_+]}$$

Usually the threshold L is chosen lower than the expected portfolio returns. In the portfolio insurance framework, additional constraints must be taken into account. For example, the threshold must be higher than the guaranteed amount.

This subsection is dedicated to computing the Omega measure of the CPPI terminal portfolio value. Recall that the terminal CPPI portfolio value $V_{T+} = C_{T+} + F_T$ and therefore, the discounted terminal value

$$V_{T+}^* = \frac{V_{T+}}{F_T} = \frac{C_{T+} + F_T}{F_T} = C_{T+}^* + 1, \quad (41)$$

which reduces to 1 when the terminal portfolio value V_{T+} is at a guarantee value F_T . Consequently, it is prudent to consider the following Omega measure at certain constant threshold $L \geq 1$ for the CPPI portfolio:

$$\Omega(V_{T+}^* | H_0) = \frac{\mathbb{E} \left[(V_{T+}^* - L)_+ | H_0 \right]}{\mathbb{E} \left[(L - V_{T+}^*)_+ | H_0 \right]} = \frac{\mathbb{E} \left[(C_{T+}^* - (L - 1))_+ | H_0 \right]}{\mathbb{E} \left[((L - 1) - C_{T+}^*)_+ | H_0 \right]}. \quad (42)$$

When $L = 1$, the Omega measure is given by

$$\Omega(V_{T+}^* | H_0) = \frac{\mathbb{E} \left[(C_{T+}^*)_+ | H_0 \right]}{\mathbb{E} \left[(-C_{T+}^*)_+ | H_0 \right]} = \frac{\text{UEG}_{H_0}}{\text{UES}_{H_0}},$$

where UEG_{H_0} and UES_{H_0} have been derived analytically in Propositions 3.2 and 3.3, respectively.

Next, we assume $L > 1$. By denoting $d = (L - 1)/C_{t_0}^*$, we can rewrite the Omega measure as follows

$$\Omega(V_{T+}^*, d | H_0) = \frac{\mathbb{E} \left[(C_{T+}^*/C_{t_0}^* - d)_+ | H_0 \right]}{\mathbb{E} \left[(C_{T+}^*/C_{t_0}^* - d)_+ | H_0 \right] + d - \mathbb{E} \left(C_{T+}^*/C_{t_0}^* | H_0 \right)}, \quad (43)$$

where d in $\Omega(V_{T+}^* | H_0, d)$ is used to signify the dependence of the Omega measure on the quantity d . The item $\mathbb{E} \left(C_{T+}^*/C_{t_0}^* | H_0 \right)$ in the denominator of the right hand side of (43) can be computed as follows:

$$\mathbb{E} \left(C_{T+}^*/C_{t_0}^* | H_0 \right) = (\text{UEG}_{H_0} - \text{UES}_{H_0})/C_{t_0}^*. \quad (44)$$

Therefore, it remains to compute $\mathbb{E} \left[(C_{T+}^* - d)_+ | H_0 \right]$ in order to obtain the value of the Omega measure.

Directly computing $\mathbb{E} \left[(C_{T+}^*/C_{t_0}^* - d)_+ | H_0 \right]$ seems technically impossible. We resort to a Laplace transform methodology. We write $u = -\ln d$ to obtain

$$\mathbb{E} \left[(C_{T+}^*/C_{t_0}^* - d)_+ | H_0 \right] = A(u | H_0), \quad (45)$$

where

$$\begin{aligned} A(u | H_0) &= \mathbb{E} \left[(e^Z - e^{-u})_+ \mathbb{I}_{\{C_{T+}^* > 0\}} | H_0 \right] \\ &= \mathbb{E} \left[(e^Z - e^{-u})_+ | C_{T+}^* > 0 | H_0 \right] \cdot \Pr(C_{T+}^* > 0 | H_0), \quad u \in \mathbb{R}, \end{aligned}$$

and $Z = \ln \left(C_{T+}^* / C_{t_0^+}^* \right)$. Consequently, combining (43) in (44) and (45) yields

$$\Omega(V_{T+}^*; d|H_0) = \frac{A(u|H_0)}{A(u|H_0) + e^{-u} - (\text{UEG}_{H_0} - \text{UES}_{H_0}) / C_{t_0^+}^*},$$

with $u = -\ln d$. (46)

Let i denote the complex unity so that $(-i)^2 = -1$. For $\sigma > 0$, the double-sided Laplace transform of $A(u)$ can be computed as follows:

$$\begin{aligned} \mathcal{L}_A(\sigma + i\omega|H_0) &= \int_{-\infty}^{\infty} e^{-(\sigma+i\omega)u} A(u) du \\ &= \int_{-\infty}^{\infty} e^{-(\sigma+i\omega)u} \mathbb{E} \left[(e^Z - e^{-u})_+ \middle| C_{T+}^* > 0 \right] \Pr(C_{T+}^* > 0) du \\ &= \int_{-\infty}^{\infty} e^{-(\sigma+i\omega)u} \int_{-u}^{\infty} (e^z - e^{-u}) f_{Z_+}(z) dz du \cdot \Pr(C_{T+}^* > 0) \\ &= \int_{-\infty}^{\infty} \left[f_{Z_+}(z) \cdot \Pr(C_{T+}^* > 0) \int_{-z}^{\infty} e^{-(\sigma+i\omega)u} (e^z - e^{-u}) du \right] dz \\ &= \left(\frac{1}{\sigma + i\omega} - \frac{1}{1 + \sigma + i\omega} \right) \int_{-\infty}^{\infty} e^{iz[\omega - i(1+\sigma)]} f_{Z_+}(z) \cdot \Pr(C_{T+}^* > 0) dz \\ &= \frac{1}{[\sigma + \sigma^2 - \omega^2] + i\omega[2\sigma + 1]} \mathcal{C}_+(\omega - i(1 + \sigma) | H_0), \end{aligned} \tag{47}$$

where $f_{Z_+}(z)$ denotes the density function of Z conditional on the event of $\{C_{T+}^* > 0\}$, and

$$\mathcal{C}_+(u|H_0) = \mathbb{E} \left[e^{iuZ} \mathbb{I}_{\{C_{T+}^* > 0\}} \middle| H_0 \right]. \tag{48}$$

An explicit formula of $\{C_{T+}^* > 0\}$ is given in Proposition 3.4 below.

Proposition 3.4. Define $\boldsymbol{\psi}^+(u) = (\psi_1^+(u), \dots, \psi_r^+(u))'$ and $\boldsymbol{\phi}^+(u) = (\phi_1^+(u), \dots, \phi_r^+(u))'$, where $\psi_j^+(u) = \ln \mathbb{E} \left[e^{iu \ln \Xi_k^{(j)}} \mathbb{I}_{\{\Xi_k^{(j)} > 0\}} \right]$, $\phi_j^+(u) = \ln \mathbb{E} \left[e^{iu \ln \Lambda_k^{(j)}} \mathbb{I}_{\{\Lambda_k^{(j)} > 0\}} \right]$, $j = 1, \dots, r$, and by convention $\ln x = \ln |x| + \arg(x)$ for a complex number x . Further denote

$$\mathcal{D}_{\boldsymbol{\phi}^+}(u) = \text{diag} \left(e^{\phi_1^+(u)}, \dots, e^{\phi_r^+(u)} \right) \quad \text{and} \quad e^{\boldsymbol{\psi}^+(u)} = \left(e^{\psi_1^+(u)}, \dots, e^{\psi_r^+(u)} \right)'.$$

The functions $\mathcal{C}_+(u|H_0)$ defined in (48) can be computed explicitly as follows:

$$\mathcal{C}_+(u|H_0) = \left(e^{\boldsymbol{\psi}^+(u)} \right)' \mathbf{P} \left(\mathcal{D}_{\boldsymbol{\phi}^+(u)} \mathbf{P} \right)^{n-1} H_0, \tag{49}$$

Proof. Recall the expression of C_{T+}^* in (21), and note that $\{C_{T+}^* > 0\} = \{\varrho > n\}$. Also note the fact that $\{(\Lambda_t, \Xi_t), t = 1, \dots, k\}$ are independent and identically distributed random vectors

given \mathcal{F}_n^H . Thus,

$$\begin{aligned}
\mathbb{E} \left[e^{iuZ} \mathbb{I}_{\{C_{T^+}^* > 0\}} \middle| \mathcal{F}_n^H \right] &= \mathbb{E} \left[e^{iu \ln \Xi_n} \mathbb{I}_{\{\Xi_n > 0\}} \cdot \prod_{j=1}^{n-1} (e^{iu \ln \Lambda_j} \mathbb{I}_{\{\Lambda_j > 0\}}) \middle| \mathcal{F}_n^H \right] \\
&= \mathbb{E} \left[e^{iu \ln \Xi_n} \mathbb{I}_{\{\Xi_n \leq 0\}} \middle| H_n \right] \cdot \prod_{j=1}^{n-1} \mathbb{E} \left[e^{iu \ln \Lambda_j} \mathbb{I}_{\{\Lambda_j > 0\}} \middle| H_j \right] \\
&= \exp \left\{ \psi_{E_n}^+(u) + \sum_{j=1}^{n-1} \phi_{E_j}^+(u) \right\},
\end{aligned}$$

where $E_j = i$ if $H_j = h_i$ for $i = 1, \dots, r$ and $j = 1, \dots, n$. Consequently, the desired result follows from part (a) of Lemma 3.2 as follows

$$\begin{aligned}
\mathcal{C}_+(u|H_0) &= \mathbb{E} \left[\exp \left\{ \psi_{E_n}^+(u) + \sum_{j=1}^{n-1} \phi_{E_j}^+(u) \right\} \middle| H_0 \right] \\
&= \mathbb{E} \left[\left(e^{\boldsymbol{\psi}^+(u)} \right)' H_n \cdot \exp \left\{ \sum_{j=1}^r (\Gamma_j(n-1) \cdot \phi_j^+(u)) \right\} \middle| H_0 \right] \\
&= \left(e^{\boldsymbol{\psi}^+(u)} \right)' \mathbb{E} \left[H_n \cdot \exp \left\{ \sum_{j=1}^r (\Gamma_j(n-1) \cdot \phi_j^+(u)) \right\} \middle| H_0 \right] \\
&= \left(e^{\boldsymbol{\psi}^+(u)} \right)' \mathbf{P} \left(\mathcal{D}_{\boldsymbol{\phi}^+(u)} \mathbf{P} \right)^{n-1} H_0.
\end{aligned}$$

□

The expression for $\mathcal{L}_A(\cdot)$ in (47) and equation (49) for $\mathcal{C}_+(\cdot|H_0)$ in Proposition (3.4) allow us to adopt certain Laplace inversion algorithm to compute the Omega measure defined in (42). In the numerical analysis in next section, a double-sided Laplace inversion algorithm recently developed by Cai et al. (2013) will be applied for this purpose.

4 Numerical analysis

A numerical example based on a dataset of S&P 500 index will be presented in this section to illustrate the implementation of those formulae of risk and reward measures developed in the previous sections.

4.1 Numerical setting

A CPPI portfolio will be studied with weekly rebalancing between a risky asset and a bond for $T = 2$ (years), i.e., $n = 104$ weeks with 52 weeks for each year. We consider an initial value of $V_0 = 1,000$ and a guarantee value of $F_T = 1,000$ for the CPPI portfolio. To simplify our analysis, we assume that the yield of the bond is constant at an annual rate of 5.2% so that its weekly yield rate is 0.1%. Such an assumption implies a starting cushion of $C_0 = V_0 - F_T \times e^{-T \times 5.2\%} = 98.7747$ and a floor $F_t = 901.2253e^{0.001t}$, $0 \leq t \leq 104$.

The S&P 500 index is chosen as the risky asset and the financial market is assumed to be governed by a Markov-modulated regime switching model with three distinct regimes: *bullish*, *normal* and *bearish* states. Under each regime, the log-return of the index is subject to a normal distribution with mean μ_i and standard deviation σ_i given as follows:

$$\mu_1 = -0.0159, \quad \mu_2 = 0.0013, \quad \mu_3 = 0.0027, \quad \sigma_1 = 0.0609, \quad \sigma_2 = 0.0254, \quad \sigma_3 = 0.0141.$$

The transition probability matrix for the regime process is as follows

$$P = (p_{i,j})_{i,j=1,2,3} = \begin{pmatrix} 0.8241, & 0.0154, & 0.0000 \\ 0.1759, & 0.9743, & 0.0069 \\ 0.0000, & 0.0103, & 0.9931 \end{pmatrix},$$

where $p_{i,j} = \Pr(\xi_t = i | \xi_{t-1} = j)$, $i, j = 1, 2, 3$. The above parameters for the risky asset are calibrated from 1,248 weekly closing data points of the S&P 500 index from January 01, 1990 throughout November 10, 2013. With the above parameter values, the first regime is a bear market state, as it has a negative expected return accompanied and a large volatility. The third regime is a bull state because it has a positive expected return and a small volatility. Compared with the third regime, the second one has a smaller expected return and a relatively larger volatility in the meanwhile; thus, the second state can be viewed as an intermediate state.

4.2 Numerical results

With the numerical setting in the preceding subsection, the shortfall probability, the unconditional expected shortfall and the unconditional expected gain can be computed by their formulae given in Propositions 3.1, 3.2 and 3.3, respectively. The resulting shortfall probability is showed in Figure 1 for distinct combination of the starting market state H_0 , the transaction cost level ϱ and the multiplier m .

The expected dollar amount of the risk on the guarantor at the terminal transaction time T is given by

$$\mathbb{E} [C_{T+} \mathbb{I}_{\{\varrho \leq n\}}] = F_T \cdot \mathbb{E} [C_{T+}^* \mathbb{I}_{\{\varrho \leq n\}}] = 1,000 \times \text{UES}_{H_0}.$$

We apply the formula (33) for UES_{H_0} and show the value of $\mathbb{E}[C_{T+} | \varrho \leq n]$ in Figure 2. Similarly, the expected value of the terminal payoff for the investor is given by

$$\mathbb{E} [C_{T+} \mathbb{I}_{\{\varrho > n\}}] = F_T \cdot \mathbb{E} [C_{T+}^* \mathbb{I}_{\{\varrho > n\}}] = 1000 \times \text{UEG}_{H_0}$$

We apply the formula (40) to compute UEG_{H_0} and show the value of $\mathbb{E} [C_{T+} \mathbb{I}_{\{\varrho > n\}}]$ in Figure 3. Moreover, we apply formula (46) to compute the Omega measure of V_+^* at a series of different threshold d and show the results in Figure 4. In the specific implementation, the values of UES_{H_0} and UEG_{H_0} in the formula (46) are computed by their formulae given in (33) and (40), and the value of $A(u|H_0)$ is obtained from the Laplace transform (47) by the double-side Laplace inversion algorithm from Cai et al. (2013) with algorithm parameters $\sigma = 0.3$, $C = 10$ and $N = 350$.

In Figures 1-4, the three (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively denote the bearish, the intermediate and the bullish initial market state as described in the preceding subsection. For presentation convenience, the CPPI portfolio with the three different initial market states will be

respectively referred to as “bearish-start portfolio”, “normal-start portfolio” and “bullish-start portfolio”. The results in Figures 1-4 show that the regime switching feature has a significant effect on the performance of the CPPI portfolio. Different market state at the inception of the investment leads to quite distinct risk-and-reward profile for the CPPI portfolio. More detailed comments are given under each of the Figures 1-4.

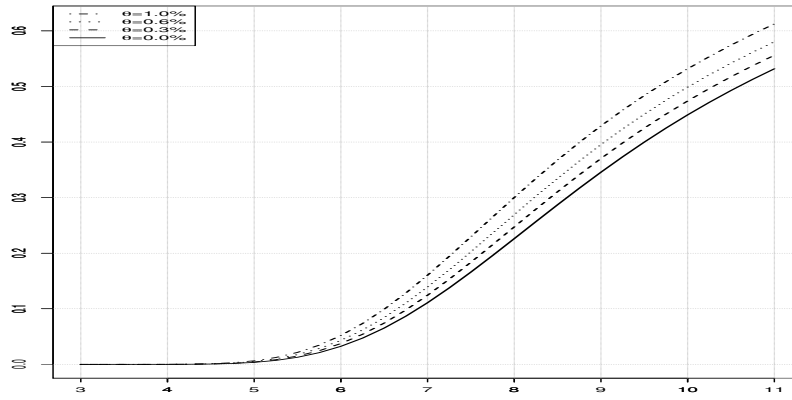
Moreover, Figures 1-4 also illustrate the significant effect from the transaction cost on each risk and/or reward measures of the CPPI portfolio. More detailed comments about its effects on the shortfall probability, the unconditional expected shortfall and the Omega measure are given beneath the Figures 1, 2 and 4 respectively. Its effect on the unconditional expected gain is the most intricate. Figure 3 indicates: the higher the transaction costs are, the lower the investor’s unconditional expected gain is. This confirms the unsurprising fact that the transaction cost hampers the investor’s returns. More interestingly, the transaction cost changes the effect pattern of the multiplier on the unconditional expected gain. When there is no transaction cost, it has been widely recognized that the expected gain of a CPPI portfolio is increasing in the multiplier m , as confirmed by plot a) of Figure (3). Nevertheless, when the transaction is not cost free, the expected gain is not necessarily increasing all the way with the multiplier m , as indicated by the other three plots of Figure (3). In particular, when the transaction cost proportional factor θ is as large as 1%, the expected gain for both the bearish-start portfolio and the normal-start portfolio is decreasing in m over all the value of m considered, and it is increasing with m up to 6 and then decreasing all the way thereafter.

5 Concluding Remark

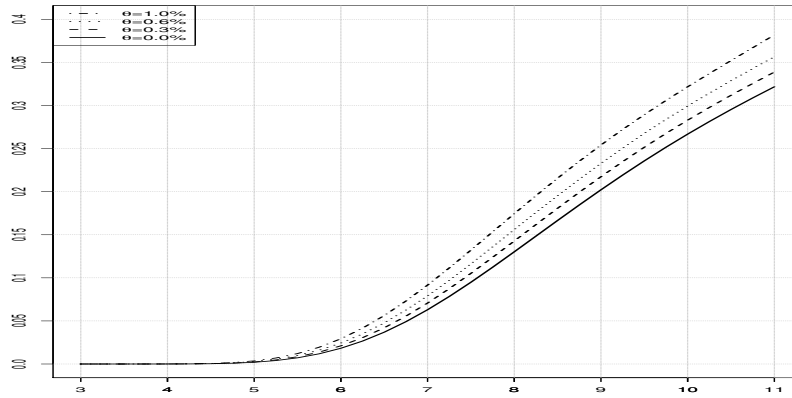
The regime switching framework for modeling econometric series offers a transparent and intuitive way to capture market behavior through different economic conditions. It has been widely used in econometrics since the pioneering work of Hamilton (1989). The present paper developed a framework for studying the performance of discrete-time CPPI portfolio in the presence of proportional transaction cost and regime switching. Analytically tractable expressions for the shortfall probability and the expected shortfall of the “gap risk” and the expected gain of the investor’s payoff are derived under a general Markov-modulated regime switching model. A double-sided Laplace inversion method is developed for computing the Omega measure of the discrete-time CPPI portfolio. The implementation of these results are illustrated by a numerical example with a real data set of S&P 500 index. The numerical example shows that the behavior of the CPPI portfolio can differ significantly with different market state at the inception of the investment. The results established in this paper can not only help the investors of a CPPI fund to develop a comprehensive understanding on their risk-and-reward profile, but also offer an effective framework for CPPI fund guarantors to conduct stress tests.

Acknowledgements

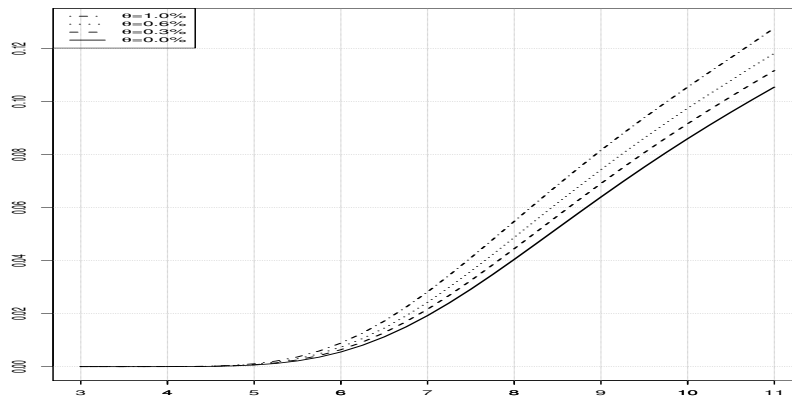
The author is grateful to the support from the Natural Sciences and Engineering Research Council of Canada (NSERC, No. 368474) and Society of Actuaries Centers of Actuarial Excellence Research Grant.



a) $H_0 = (1, 0, 0)$, starting from a bearish market

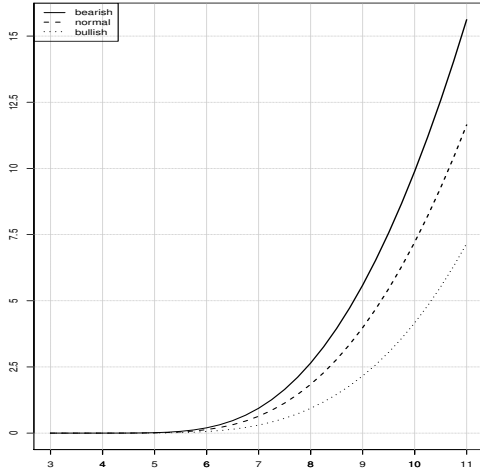


b) $H_0 = (0, 1, 0)$, starting from a normal market

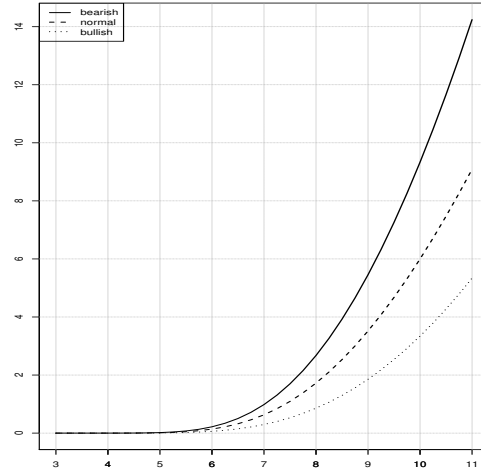


c) $H_0 = (0, 0, 1)$, starting from a bullish market

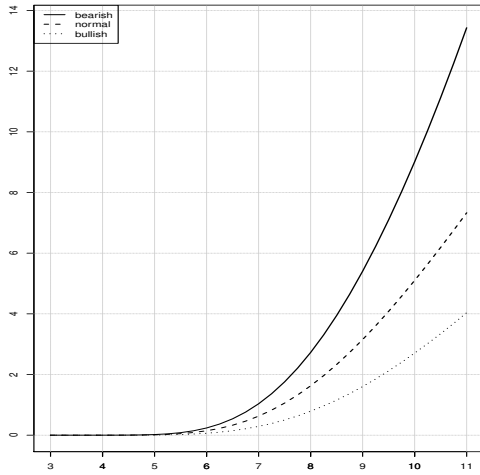
Figure 1: This figure shows the shortfall probability as a function of the multiplier m for a CPPI portfolio starting from three different market states and at four distinct levels of transaction cost. At any of the four levels of transaction cost, the shortfall probability is substantially larger for a portfolio starting from a bearish initial market state than any of the other two states. For example, as indicated by plot a), there is a probability of more than 20% to realize a shortfall within the investment horizon for a CPPI portfolio with $m = 8$ and $\theta = 0$ and a bearish initial market state. In contrast, the probability is only about 4% (13%) if the portfolio starts in a bullish (normal) market. Moreover, the figure also shows that the higher the transaction cost, the higher the shortfall probability, which is in accordance to our intuition that the transaction cost reduces the portfolio value and therefore increases the chances for the portfolio value to drop below the floor.



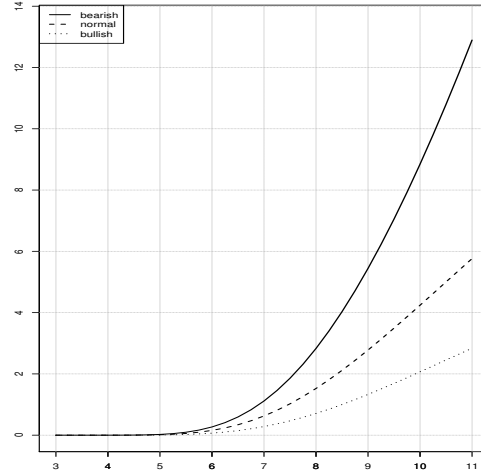
a) $\theta = 0.0\%$



b) $\theta = 0.3\%$

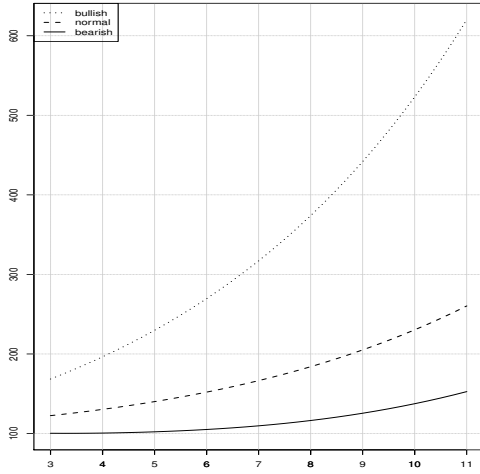


c) $\theta = 0.6\%$

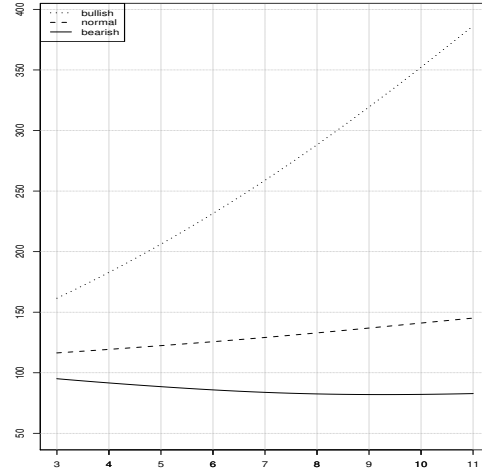


d) $\theta = 1.0\%$

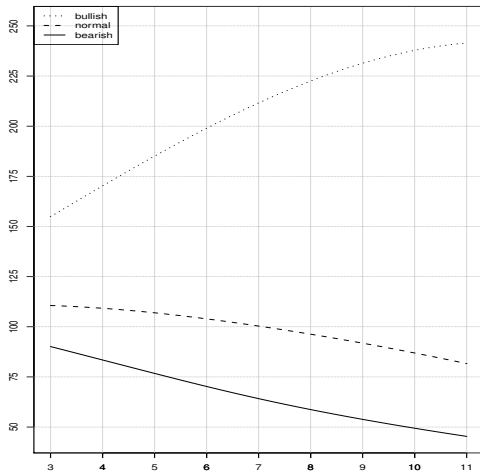
Figure 2: This figure shows the unconditional expected shortfall (UES) as a function of the multiplier m for the CPPI portfolio starting from three different initial market states and at four distinct levels of transaction cost. The UES is obviously larger for a bearish-start portfolio than a portfolio starting from either of the other two states, regardless of transaction costs. This implies that the risk on the CPPI fund guarantor substantially differs from one initial market state to another. There is more risk on the guarantor if the investor enters the fund in a bear market than a bull market. Compared these four plots, we see that the existence of the transaction cost indeed lessen the risk on the fund guarantor, as it decreases the unconditional expected shortfall, though very slightly. At the first thought, it seems a contradiction to our intuition. Nevertheless, it indeed accords with the CPPI strategy well. The existence of the transaction cost reduces the cushion value over each portfolio revision, and therefore the exposure in the risky asset is also reduced accordingly. This, in turn, alleviates the shortfall level once a gap risk occurs.



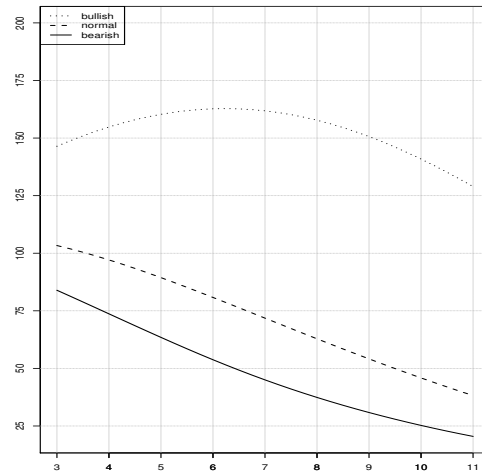
a) $\theta = 0.0\%$



b) $\theta = 0.3\%$

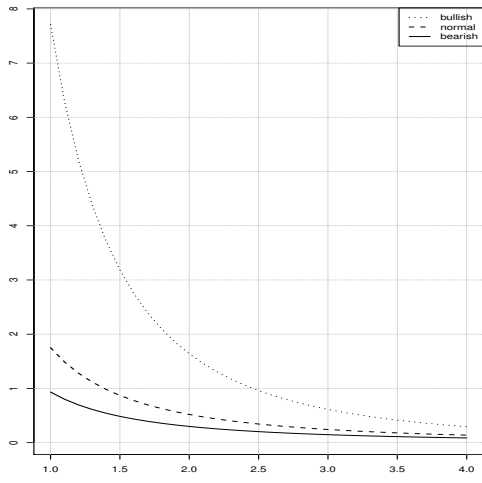


c) $\theta = 0.6\%$

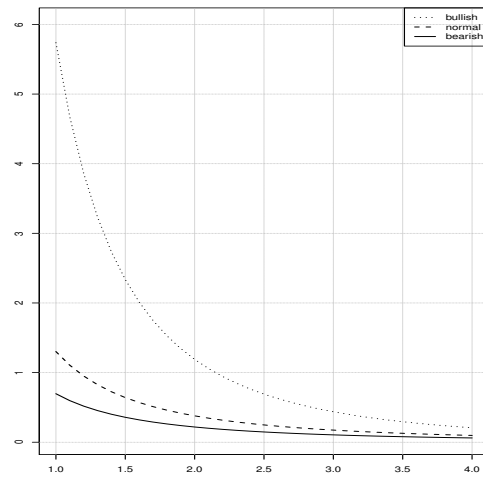


d) $\theta = 1.0\%$

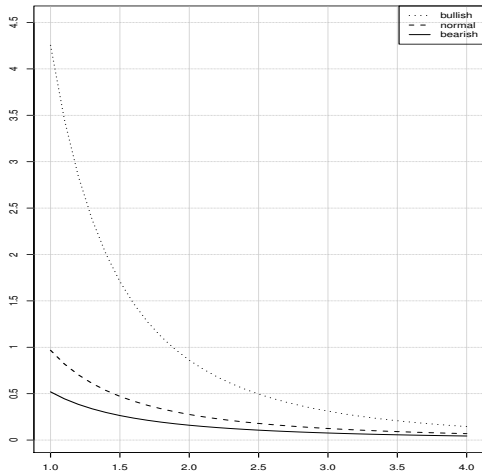
Figure 3: This figure shows the unconditional expected gain as a function of the multiplier m for the CPPI portfolio starting from three different market states and at four distinct levels of transaction cost. For all of the four transaction cost levels, the investors should expect to receive a much higher unconditional expected gain if they start their portfolio in a bullish market than a normal or a bearish market.



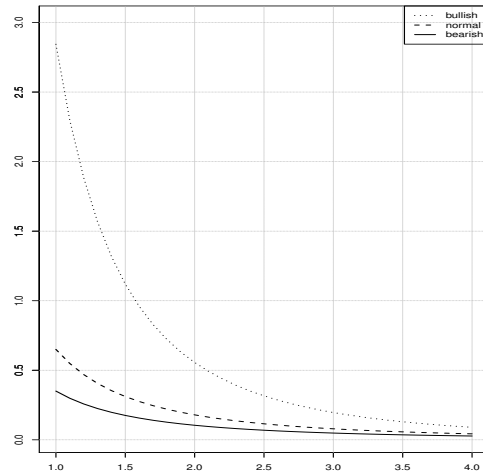
a) $\theta = 0.0\%$



b) $\theta = 0.3\%$



c) $\theta = 0.6\%$



d) $\theta = 1.0\%$

Figure 4: This figure shows the Omega measure of the terminal portfolio value as a function of the threshold d for CPPI portfolios starting from three different market states and at four distinct levels of transaction cost: $m = 6$ and $T = 2$. These four plots confirm an order among the three portfolios starting from different market states with respect to their performance. The bullish-start one is the best, and the bearish one is the worst. This figure also consistently shows a negative effect the transaction cost exerts on the performance of the CPPI portfolio. For all the three starting market states, an increase in transaction costs substantially reduces the Omega measure.

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CHENGGUO WENG (Corresponding author)
Department of Statistics and Actuarial Science,
M3-200 University Avenue West,
University of Waterloo,
Waterloo, Ontario, N2L 3G1, Canada.
E-Mail: c2weng@uwaterloo.ca